WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 541, August, 2016

Prove all results. You may quote standard named results proved in class.

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf $f(x|\theta) = e^{-(x-\theta)} I_{(\theta,\infty)}(x)$, where $-\infty < \theta < \infty$. Show that $X_{(1)} = \min_{1 \le i \le n}\{X_i\}$ and the sample variance S^2 are independent.

2. Suppose $X_1, \cdots, X_n$ is a random sample from a binomial distribution with parameters N and θ, i.e., Bin(N, θ), where N is a positive integer and $0 < \theta < 1$.

   (a) Determine the maximum likelihood estimator $\hat{\theta}$ for θ using the X_i.

   (b) Show
   $$\tilde{\theta} = \frac{\bar{X}}{\max\{X_1, \cdots, X_n\}}$$
   is a consistent estimator of θ.

   (c) For large n, which estimator $\hat{\theta}$ or $\tilde{\theta}$, would you prefer? Justify.

3. As in the previous problem, suppose $X_1, \cdots, X_n$ is a random sample from a binomial distribution with parameters N and θ, i.e., Bin(N, θ), where N is a positive integer and $0 < \theta < 1$.

   (a) Find an approximate 95% confidence interval for θ.

   (b) Consider the hypothesis:
   $$H_0 : \theta \le \theta_0, \quad v.s. \quad H_1 : \theta > \theta_0,$$
   for some specified value θ_0. Give the uniformly most powerful test at the α level of significance. Justify.

4. Let $X_1, \ldots, X_n \mid \mu, \sigma^2 \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$. Let $\bar{X}$ denote the sample mean and S^2 the sample variance. Find $f(x_1, \ldots, x_n \mid \bar{x}, s^2, \mu, \sigma^2)$. Are there surprises? Explain.

5. Let $Y_i \overset{ind}{\sim} \text{Bernoulli}\{\frac{e^{x_i\beta}}{1+e^{x_i\beta}}\}$, where $x_1, \ldots, x_n$ are fixed known real numbers. What conditions are needed for the existence of the MLE of β? Write down an approximate distribution for the MLE. Find an approximate 95% confidence interval for β. If you were able to find the exact interval, would it be shorter or longer than the approximate one? Explain.